Momento angular orbital $\hat{\vec{L}} = \hat{\vec{R}} \times \hat{\vec{P}}$

- Usando la representación $\{|\vec{r}\rangle\}$

$$\hat{\vec{R}} \to \vec{r} \qquad \hat{\vec{P}} \to -i\hbar\vec{\nabla} \qquad \int |\psi|^2 dV = 1$$

En coordenadas esféricas $\psi(\vec{r}) = \psi(r, \theta, \varphi)$

$$L_z = -i\hbar \frac{\partial}{\partial \varphi}$$

$$L^2 = -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan\theta} \frac{\partial}{\partial \theta} + \frac{1}{\operatorname{sen}^2\theta} \frac{\partial^2}{\partial \varphi^2} \right)$$

$$L^2 |l\, m\rangle = \hbar^2 l(l+1) |l\, m\rangle \xrightarrow{\{|\vec{r}\rangle\}} -\left[\frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan\theta} \frac{\partial}{\partial \theta} + \frac{1}{\operatorname{sen}^2\theta} \frac{\partial^2}{\partial \varphi^2} \right] \psi(r,\theta,\varphi) = l(l+1)\psi(r,\theta,\varphi)$$

$$L_z |l\, m\rangle = \hbar m |l\, m\rangle \qquad -i \frac{\partial}{\partial \varphi} \psi(r,\theta,\varphi) = m\psi(r,\theta,\varphi).$$

- Para l fijo los valores permitidos de m son $m = -l, -l+1, \ldots, l-1, l$

- Como r no aparece en los operadores diferenciales al hacer separación de variables $\psi(r,\theta,\varphi) = f(r) Y_l^m(\theta,\varphi)$ obtenemos

$$L^2 Y_l^m(\theta,\varphi) = \hbar^2 l(l+1) Y_l^m(\theta,\varphi)$$

$$L_z Y_l^m(\theta,\varphi) = \hbar m Y_l^m(\theta,\varphi)$$

(Sin dependencia angular)

$$-i \frac{\partial}{\partial \varphi} Y_l^m(\theta,\varphi) = m Y_l^m(\theta,\varphi)$$

Usando separación de variables

$$Y_l^m(\theta,\varphi) = F_l^m(\theta)\Phi(\varphi)$$

no ponemos constante pues se la achacamos a la $F_l^m(\theta)$

$$\Rightarrow -i\Phi'(\varphi) = m\Phi(\varphi) \Rightarrow \Phi(\varphi) = e^{im\varphi}$$

$$\therefore\ Y_\ell^m(\theta,\varphi) = F_\ell^m(\theta)\, e^{im\varphi}$$

Veamos que para el momento angular orbital, ℓ debe ser entero positivo.
(i.e. en este caso no puede tener un valor semi-entero).

Para que $Y_\ell^m(\theta,\varphi)$ sea univaluada.

$$Y_\ell^m(\theta,\varphi) = Y_\ell^m(\theta, \varphi + 2\pi)$$

$$\Rightarrow \quad \cancel{F_\ell^m(\theta)}\, \cancel{e^{im\varphi}} = F_\ell^m(\theta)\, e^{im(\cancel{\varphi} + 2\pi)}$$

$$\Rightarrow \quad 1 = e^{im2\pi}$$

$\Rightarrow$ m es entero

$\Rightarrow$ ℓ es entero.

Una forma de obtener las soluciones sin usar la ecuación diferencial espantosa,

$$-\left[\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\operatorname{sen}^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] Y_\ell^m(\theta,\varphi) = \ell(\ell+1)\, Y_\ell^m(\theta,\varphi)$$

es utilizar la teoría de momento angular que ya tenemos.

Por ejemplo, sabemos que $L_+ Y_\ell^{m=\ell} = 0$

$\hat{L}_+ |\ell, m=\ell\rangle = 0$

$$L_x = i\hbar\left(\operatorname{sen}\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right)$$

$$L_y = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \frac{\operatorname{sen}\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right)$$

$e^{i\varphi}$ $\qquad$ $e^{i\varphi}$

$$L_+ = L_x + i L_y$$

$$= \hbar\left[i \operatorname{sen}\varphi \frac{\partial}{\partial\theta} + i\frac{\cos\varphi}{\tan\theta}\frac{\partial}{\partial\varphi} + \cos\varphi\frac{\partial}{\partial\theta} - i\frac{\operatorname{sen}\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right]$$

$$L_+ = \hbar e^{i\varphi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right]$$

$$L_- = \hbar e^{-i\varphi}\left[-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right]$$

Si resolvemos la ecuación

$$L_+ Y_\ell^{m=\ell}(\theta,\varphi) = 0$$

para obtener Y_ℓ^ℓ, podemos usar L_- para obtener $Y_\ell^m(\theta,\varphi)$ para los $m<\ell$.

$m=\ell$

$$L_+ Y_\ell^{m=\ell}(\theta,\varphi)=0 \Rightarrow \cancel{\hbar}\,\cancel{e^{i\varphi}}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right]F_\ell^m(\theta)\,e^{i\ell\varphi} = 0$$

$$\left[\frac{\partial}{\partial\theta} - \ell\cot\theta\right]F_\ell^m(\theta) = 0$$

$$\frac{1}{F_\ell^m(\theta)}\frac{dF_\ell^m(\theta)}{d\theta} = \ell\,\frac{\cos\theta}{\operatorname{sen}\theta}$$

$$\Rightarrow \frac{d}{d\theta}\ln\left(F_\ell^m(\theta)\right) = \ell\,\frac{d}{d\theta}\ln(\operatorname{sen}\theta)$$

integrando respecto a θ

$$\ln F_\ell^m(\theta) = \ell\ln(\operatorname{sen}\theta) + C$$

$$F_\ell^m(\theta) = \underbrace{e^C}_{N}\operatorname{sen}^\ell(\theta) = N\operatorname{sen}^\ell\theta$$

$$\therefore \quad Y_l^{m=l}(\theta,\varphi) = F_l^{m=l}(\theta)\,\Phi(\varphi) = N \operatorname{sen}^l\theta \; e^{im\varphi}$$

Vamos a escoger N de tal manera que

$$\int_0^{2\pi}\int_0^{\pi} |Y_l^m(\theta,\varphi)|^2 \operatorname{sen}\theta \, d\theta \, d\varphi = 1$$

para que

$$\int |\psi(\vec{r})|^2 dV = \int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} |f(r)|^2 \, |Y_l^m(\theta,\varphi)|^2 \, r^2 \operatorname{sen}\theta \, d\theta \, d\varphi \, dr$$

$$= \left[\int_0^{\infty} |f(r)|^2 r^2 \, dr\right]\left[\int_0^{2\pi}\int_0^{\pi} |Y_l^m(\theta,\varphi)|^2 \operatorname{sen}\theta \, d\theta \, d\varphi\right]$$

$l=0$	$l=1$	$l=2$
$Y_0^0 = \sqrt{\frac{1}{4\pi}}$	$Y_1^{\pm 1} = \pm\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi}$ $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$	$Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\varphi}$ $Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\varphi}$ $Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$

La expresión general para cualquier valor de l y m está dada por

$$Y_l^m(\theta,\varphi) = (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi}, \quad m \geq 0;$$

$$Y_l^{-m}(\theta,\varphi) = (-1)^m Y_l^m(\theta,\varphi)^*;$$

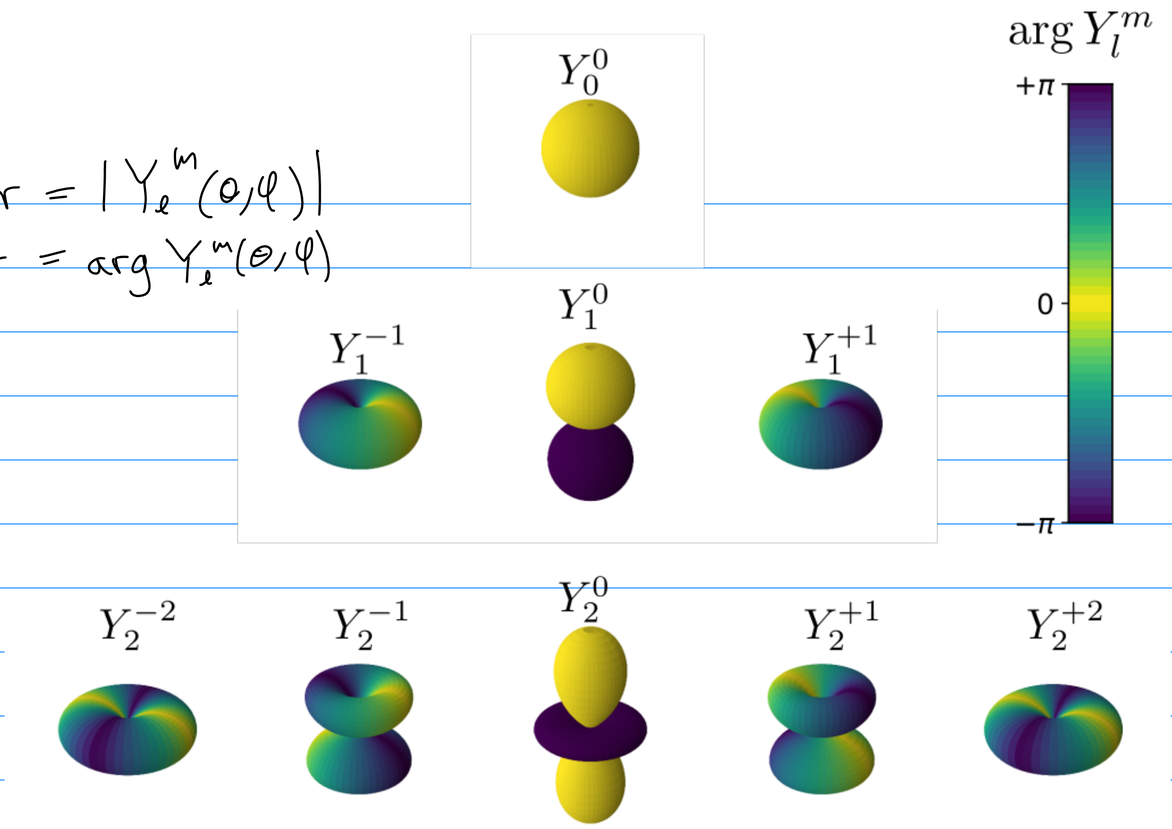
$$P_l^m(x) = (1-x^2)^{|m/2|}\frac{d^{|m|}}{dx^{|m|}}P_l(x);$$

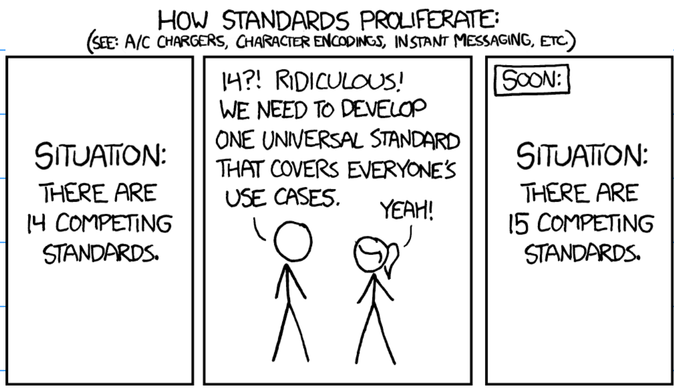
$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l.$$

Aquí, a $P_l^m(x)$ se le conoce como polinomio asociado de Legendre.

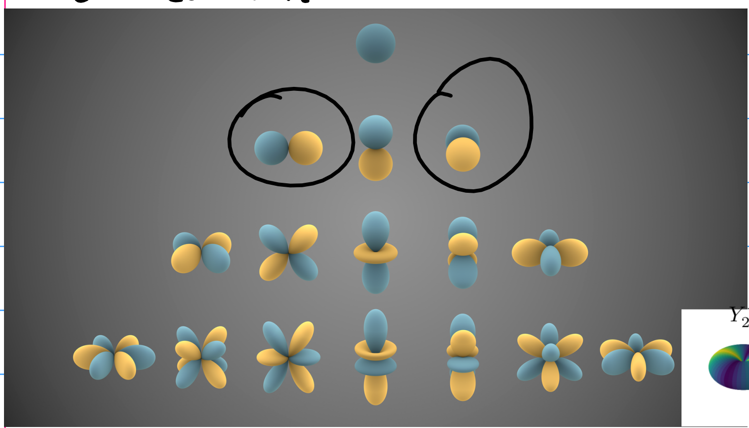
$r = |Y_\ell^m(\theta,\varphi)|$

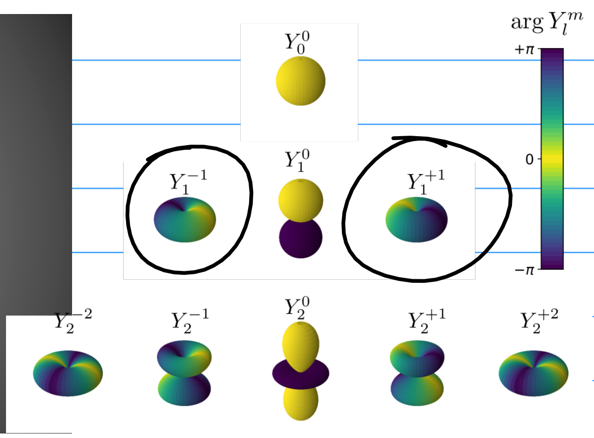
$\text{color} = \arg Y_\ell^m(\theta,\varphi)$

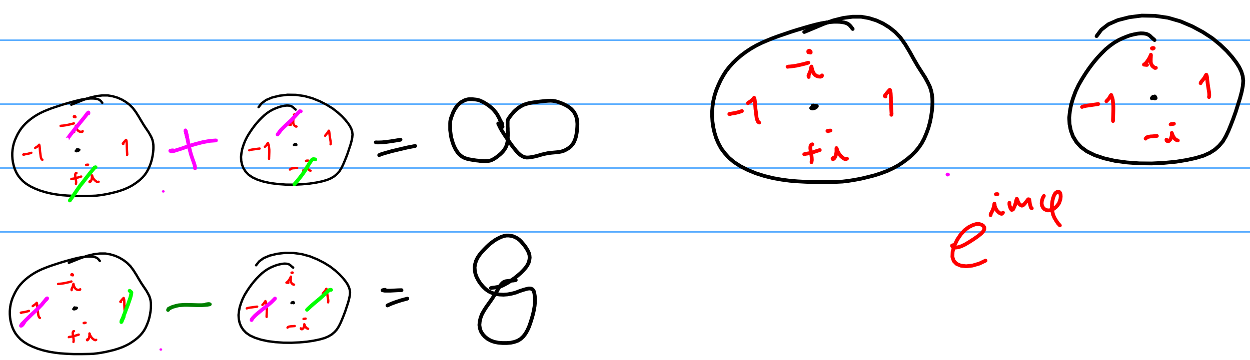




Convención real, no se usa tanto en cuántica.







$e^{im\varphi}$